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Short Communication

New fluid velocity expression in an extensible semi-circular pipe conveying fluid

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Abstract

A new expression for fluid velocity is presented for a fluid-conveying semi-circular pipe with an extensible centreline. The proposed fluid velocity expression is obtained from the material derivative, while the previous velocity expressions are from Love's kinematical relations. In order to show that the new fluid velocity expression is more reasonable than the previous velocity expressions, the equations of in-plane motion derived with the new expression are compared to the equations derived with the previous expressions. Furthermore, the equilibrium positions, natural frequencies and mode shapes obtained with both the new and previous expressions are analysed and discussed.

1. Introduction

The dynamic analysis of pipes conveying fluid has been a popular topic in engineering, because these pipes are widely used in various applications, e.g., refrigerators, air-conditioners, chemical plants, hydropower systems and so on. Much research about straight pipes conveying fluid has been reported. Some examples of these studies can be found in Refs. [1–4], in which the vibrations and stability of fluid-conveying pipes were analysed. Compared to straight pipes, only a few studies of curved pipes conveying fluid have been undertaken because the curved pipes require more complicated formulation and analyses than the straight pipes. The approaches to investigate the dynamics and stability of fluid-conveying curved pipes may be classified into two types: the *inextensible* and *extensible* theories. The inextensible theory has the assumption that the centreline of a curved pipe is not stretched. On the other hand, in the extensible theory, a curved pipe can possess an extensible centreline. Misra et al. [5,6] compared the differences between the inextensible and extensible theories. They pointed out the fact that the extensible theory is more reasonable than the inextensible theory. Based on the extensible theory, Dupuis and Rousselet [7] as well as Doll and Mote [8] also derived the equations of motion for curved pipes conveying fluid. In the above papers for the extensible curved pipes, it

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should be noted that the fluid velocity was derived by using Love's kinematical relations [9]. This velocity expression is valid only for the inextensible theory. Therefore, this expression should be modified if it is used to derive the equations of motion for an extensible curved pipe transporting fluid. Otherwise, some misleading results may be obtained from the analysis of extensible pipes.

In this study, a new expression for the fluid velocity in a fluid-conveying semi-circular pipe is presented when the centreline of the semi-circular pipe is extensible. For simplicity of discussion, only the in-plane vibration is considered for the semi-circular pipe with clamped-clamped boundary conditions. The Euler–Bernoulli beam theory is adopted under the assumption of a slender pipe and the von Karman strain theory is used to consider the geometric nonlinearity. To derive the equations of motion, the extended Hamilton principle [10] is applied because the transport of fluid mass exists across the pipe boundaries. By applying three types of approaches, i.e., (1) the Hamiltonian approach with the new velocity expression, (2) the Newtonian approach with the previous velocity expression, and (3) the Hamiltonian approach with the previous velocity, the equations of motion are derived for the semi-circular pipe. In order to show that the proposed fluid velocity expression is more reasonable than the previous velocity expression, the equations of motion for the three cases are theoretically reviewed. Furthermore, the equilibrium positions, natural frequencies and mode shapes of the pipe obtained from the proposed expression of velocity are compared to those obtained from the previous velocity expression.

2. Equations of motion

Fig. 1(a) shows the top view of a slender semi-circular pipe conveying fluid, which is clamped at both ends. The semi-circular pipe has a radius of the centreline, R, and the centreline is extensible. The flow is assumed to be a plug flow with constant velocity U. The XY coordinate system is a space-fixed inertial frame and the θ coordinate is measured from the X-axis. The cross-section of the pipe is shown in Fig. 1(b) where the xyz coordinate system is a local coordinate system. In Fig. 1, $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors moving along with a fluid particle in the centreline while $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors fixed in the space.

The pipe is modelled as an Euler–Bernoulli beam, of which the planar cross-section perpendicular to the y-axis remains a plane after deformation. In order to simplify discussion in this study, it is assumed that the vibration of the pipe occurs in the XY plane. This means that only the in-plane vibration is considered in this study. In this circumstance, the motion of a point in the pipe can be represented by the displacements in the r and θ directions, u_r and u_θ:

$$u_r(x,\theta,t) = u(\theta,t), \quad u_\theta(x,\theta,t) = v(\theta,t) + x\phi(\theta,t), \tag{1}$$

where t is time, u and v are the radial and circumferential displacements of a point on the centreline, ϕ is the rotation angle about the z-axis due to the in-plane bending deformation. This angle can be expressed in terms of u and v:

$$\phi = (v - u')/R, \tag{2}$$

where the superscripted prime denotes partial differentiation with respect to θ [11].

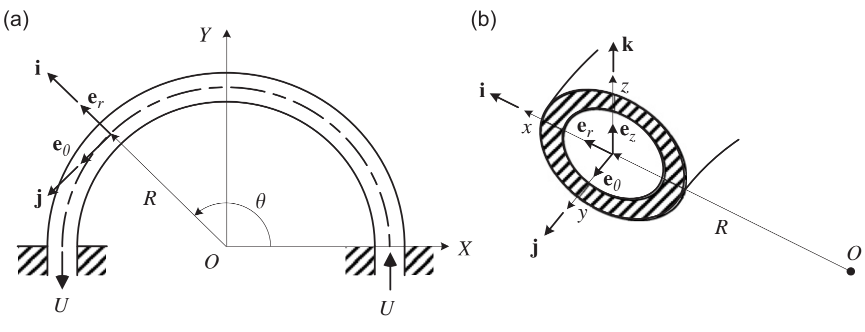

Fig. 1. Semi-circular pipe conveying fluid with velocity U: (a) top view and (b) cross-section.

To obtain the kinetic energy of the fluid-conveying semi-circular pipe, the pipe velocity and the fluid velocity need to be expressed in terms of the radial and circumferential displacements. Considering only the in-plane deformation, the displacement vector of a point in the pipe after deformation may be written as

$$\mathbf{r}_p = (u_r + x + R)\mathbf{i} + u_\theta \mathbf{j} + z\mathbf{k}. \tag{3}$$

Differentiation of Eq. (3) with respect to time results in the pipe velocity given by

$$\mathbf{v}_p = \bar{\mathbf{v}}_p + x\dot{\phi}\mathbf{j}, \tag{4}$$

where the superposed dot stands for differentiation with respect to time, and $\bar{\mathbf{v}}_p$ is the velocity of a point on the pipe centreline given by

$$\bar{\mathbf{v}}_p = \dot{u}\mathbf{i} + \dot{v}\mathbf{j}. \tag{5}$$

Similarly, the position vector of a point in the fluid after the pipe deformation can be represented by

$$\mathbf{r}_f = (u_r + x + R)\mathbf{e}_r + u_\theta \mathbf{e}_\theta + z\mathbf{e}_z. \tag{6}$$

The fluid velocity can be obtained by the material derivative with respect to time [12]. Denoting the fluid velocity by $\mathbf{v}_f$, it is given by

$$\mathbf{v}_f = \frac{\partial \mathbf{r}_f}{\partial t} + U\frac{\partial \mathbf{r}_f}{R\,\partial\theta}. \tag{7}$$

Since the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ move with a fluid particle, the angular velocity of these vectors is U/R. Therefore, the time derivatives of the unit vectors become

$$\dot{\mathbf{e}}_r = (U/R)\mathbf{e}_\theta, \quad \dot{\mathbf{e}}_\theta = -(U/R)\mathbf{e}_r. \tag{8}$$

Substitution of Eq. (6) into Eq. (7) using Eq. (8) leads to

$$\mathbf{v}_f = \bar{\mathbf{v}}_f + x\boldsymbol{\psi}, \tag{9}$$

where $\bar{\mathbf{v}}_f$ is the fluid velocity at the centreline of the pipe and $\boldsymbol{\psi}$ is the angular velocity of the fluid cross-section about the z-axis:

$$\bar{\mathbf{v}}_f = [\dot{u} + (U/R)(u' - v)]\mathbf{e}_r + [\dot{v} + (U/R)(u + v' + R)]\mathbf{e}_\theta, \tag{10}$$

$$\boldsymbol{\psi} = -(U/R)\phi\mathbf{e}_r + [\dot{\phi} + (U/R)(\phi' + 1)]\mathbf{e}_\theta. \tag{11}$$

If the rotary inertia effect of the pipe cross-section is neglected, the kinetic energy of the fluid-conveying semi-circular pipe can be approximated to

$$T = \frac{1}{2}\int_0^\pi (m_p\bar{\mathbf{v}}_p \cdot \bar{\mathbf{v}}_p + m_f\bar{\mathbf{v}}_f \cdot \bar{\mathbf{v}}_f)R\,\mathrm{d}\theta. \tag{12}$$

where m_p and m_f are the mass densities of the pipe and the fluid per unit pipe length, respectively.

Next, consider the potential or strain energy of the semi-circular pipe. If the pipe is assumed slender, it is regarded to be subjected to only circumferential stress. In this case, neglecting the gravity effect, the variation of the potential energy δV is given by

$$\delta V = \int_0^\pi \int_A \sigma\delta\varepsilon\, R\,\mathrm{d}A\,\mathrm{d}\theta, \tag{13}$$

where δ is the variation operator, σ is the circumferential stress, ε is the corresponding strain, and A is the cross-sectional area of the pipe. In order to derive the equations of motion equivalent to the equations presented in Ref. [6], the nonlinear strain and the linearized stress are used in this study. For the slender semi-circular pipe, the nonlinear strain, which is often called the von Karman strain, is expressed as

$$\varepsilon = \bar{\varepsilon} + \phi^2/2 + x\phi'/R, \tag{14}$$

where $\bar{\varepsilon}$ is the strain of the pipe centreline given by

$$\bar{\varepsilon} = (u + v')/R. \tag{15}$$

The linearized stress is obtained by deleting the nonlinear term from Eq. (14) and multiplying the remaining terms by Young's modulus E. This linearized stress can be written as

$$\sigma = E(\bar{\varepsilon} + x\phi_i'/R). \tag{16}$$

Introduction of Eqs. (14) and (16) to Eq. (13) yields

$$\delta V = \int_0^\pi \left[RQ(\delta\bar{\varepsilon} + \phi\,\delta\phi) + (EI/R)\phi'\,\delta\phi'\right]\mathrm{d}\theta, \tag{17}$$

where Q is the circumferential force across the pipe cross-section and I is the area moment of inertia about the z-axis:

$$Q = \int_A \sigma\,\mathrm{d}A, \quad I = \int_A y^2\,\mathrm{d}A. \tag{18}$$

Inserting Eq. (16) in the first equation of Eq. (18), the circumferential force can be represented as

$$Q = (EA/R)(u + v'). \tag{19}$$

The equations of motion for the semi-circular pipe conveying fluid and the associated boundary conditions are derived by using extended Hamilton's principle [10]:

$$\int_{t_1}^{t_2} (\delta T - \delta V - \delta M)\,\mathrm{d}t = 0, \tag{20}$$

where δM is the virtual momentum transport. This momentum transport should be considered to derive the equations of motion because the fluid can move across the pipe boundaries. For the given semi-circular pipe, the virtual momentum transport may be given by

$$\delta M = [m_f(\mathbf{v}_f \cdot \delta\mathbf{r})(U\mathbf{e}_\theta \cdot \mathbf{n})]_{\theta=0}^{\pi}, \tag{21}$$

where $\mathbf{n}$ is the outward normal vector at the boundaries. The equations of motion and the corresponding boundary conditions are obtained by substituting Eqs. (12), (17) and (21) into Eq. (20). The obtained equations of motion are nonlinear partial differential equations coupled between u and v:

$$\begin{aligned}(m_p + m_f)\ddot{u} &+ 2m_f(U/R)(\dot{u}' - \dot{v}) + m_f(U/R)^2(u'' - u - 2v') \\ &+ Q/R - [Q(u' - v)]'/R^2 + EI(u^{\mathrm{iv}} - v''')/R^4 = m_f U^2/R,\end{aligned} \tag{22}$$

$$\begin{aligned}(m_p + m_f)\ddot{v} &+ 2m_f(U/R)(\dot{u} + \dot{v}') + m_f(U/R)^2(2u' + v'' - v) \\ &- Q'/R - Q(u' - v)/R^2 + EI(u''' - v'')/R^4 = 0.\end{aligned} \tag{23}$$

The associated boundary conditions for the pipe are given by

$$u = u' = v = 0 \quad \text{at} \ \ \theta = 0,\ \pi. \tag{24}$$

3. Theoretical investigation

The objective of this section is to show that the proposed velocity expression is more reasonable than the previous velocity expression [5–8] derived by using Love's kinematical relations. If the semi-circular pipe is assumed to have only the in-plane motion, the fluid velocity of Refs. [5–8] can be written as

$$\bar{\mathbf{v}}_f = [\dot{u} + (U/R)(u' - v)]\mathbf{e}_r + (\dot{v} + U)\mathbf{e}_\theta. \tag{25}$$

Comparing Eqs. (10) and (25), it is seen that the second terms, i.e., the circumferential components are different from each other. If the pipe is inextensible, the strain of the pipe centreline $\bar{\varepsilon}$, given in Eq. (15), should be zero. In this case, the velocity expression of Eq. (10) becomes identical to that of Eq. (25). For this reason, it may be said that the velocity expression derived from Love's relations is valid for an inextensible pipe.

Three cases of the equations of motion for the fluid-conveying semi-circular pipe are considered in this study. The equations of motion for Case I are the equations proposed in this study. These equations are derived by using the Hamiltonian approach with the new fluid velocity expression of Eq. (10). The equations of motion for Case II are equivalent to the equations presented in Refs. [5–8], which were obtained by applying the Newtonian approach (force–moment balance) with the fluid velocity of Eq. (25). The equations of motion for Case III are derived by using the Hamiltonian approach with the velocity of Eq. (25). The differences between the three cases appear in the dynamic terms of the governing equations related to the fluid velocity. These terms correspond to the second and third terms of Eqs. (22) and (23). For these three cases, the derivation methods of the equations and the dynamic terms related to the fluid velocity are summarized in Table 1.

It should be noted that the second terms of Eqs. (22) and (23) represent the gyroscopic effects related to the fluid velocity. Therefore, if the equations of motion are discretized by an approximate method, for example, the finite element method, and if the discretized equations are expressed in a matrix-vector equation, the gyroscopic matrix should be skew-symmetric. To show that Eqs. (22) and (23) yield a skew-symmetric gyroscopic matrix after discretization, consider the weak form associated with the finite element method. Denote the weighting functions of the trial functions u and v by $\bar{u}$ and $\bar{v}$. Then, the weak form can be obtained by multiplying Eqs. (22) and (23) by $\bar{u}$ and $\bar{v}$, respectively, summing the equations, and integrating the resultant equation over $0 \leqslant \theta \leqslant \pi$. The terms representing the gyroscopic effects in the weak form may be expressed as

$$2m_f(U/R)\int_0^{\pi} \bar{\mathbf{u}}^{\mathrm{T}}\boldsymbol{\Theta}\dot{\mathbf{u}}\,\mathrm{d}\theta, \tag{26}$$

where

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \bar{\mathbf{u}} = \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix}, \quad \boldsymbol{\Theta} = \begin{bmatrix} \partial/\partial\theta & -1 \\ 1 & \partial/\partial\theta \end{bmatrix}. \tag{27}$$

Since the $\boldsymbol{\Theta}$ matrix is skew-symmetric, a gyroscopic matrix obtained by discretization becomes skew-symmetric. However, the gyroscopic terms of Cases II and III shown in Table 1 cannot produce a skew-symmetric $\boldsymbol{\Theta}$ matrix. This means that skew-symmetric gyroscopic matrices cannot be derived from the equations of Cases II and III after the equations are discretized.

Other evidence that the equations of Case I are valid can be found in Ref. [4] for a straight pipe conveying fluid. Using the notation of this paper, the dynamic terms in the equations presented in Refs. [2–4] can be expressed as

$$(m_p + m_f)\ddot{u} + 2m_f U\,\partial\dot{u}/\partial s + m_f U^2\,\partial^2 u/\partial s^2, \tag{28}$$

$$(m_p + m_f)\ddot{v} + 2m_f U\,\partial\dot{v}/\partial s + m_f U^2\,\partial^2 v/\partial s^2, \tag{29}$$

where s is the straight coordinate along the pipe centreline. When the radius of curvature for the semi-circular pipe increases to infinity, the pipe becomes a straight pipe. In this case, letting the radius R be infinite and

Table 1
Derivation methods of the equations of motion for the three cases and the dynamic terms due to the fluid velocity

Case	Derivation method	Dynamic terms related to the fluid velocity
Case I (present study)	Hamiltonian approach with the velocity of Eq. (8)	$2m_f(U/R)(\dot{u}' - \dot{v}) + m_f(U/R)^2(u'' - u - 2v')$ $2m_f(U/R)(\dot{u} + \dot{v}') + m_f(U/R)^2(2u' + v'' - v)$
Case II (Refs. [5–8])	Newtonian approach with the velocity of Eq. (23)	$2m_f(U/R)(\dot{u}' - \dot{v}) + m_f(U/R)^2(u'' - v')$ $m_f(U/R)(\dot{u} + \dot{v}') + m_f(U/R)^2(u' - v)$
Case III	Hamiltonian approach with the velocity of Eq. (23)	$m_f(U/R)(2\dot{u}' - \dot{v}) + m_f(U/R)^2(u'' - v')$ $m_f(U/R)\dot{u} + m_f(U/R)^2(u' - v)$

rewriting $R\partial\theta$ as ∂s, the dynamic terms of Eqs. (22) and (23) reduce to the terms given by Eqs. (28) and (29). However, the dynamic terms of Cases II and III cannot be transformed to Eqs. (28) and (29). Therefore, it may be concluded that the equations of Case I are more reasonable than the equations of Cases II and III.

4. Numerical investigation

In order to demonstrate the differences between the above three cases, the static equilibrium positions, the natural frequencies and the mode shapes are numerically investigated for a semi-circular pipe conveying fluid. The same solution methods presented in Ref. [6] are used in this study. In other words, solutions of the equations describing the static equilibrium and the perturbations in the neighbourhood of the static equilibrium are computed by the same finite element method formulated in Ref. [6]. The values for the physical properties and dimensions of the pipe system are $m_p = m_f = 1.78\,\text{kg/m}$, $E = 10 \times 10^9\,\text{Pa}$, $R = 0.5\,\text{m}$, $A = 2.473 \times 10^{-4}\,\text{m}^2$ and $I = 1.498 \times 10^{-7}\,\text{m}^4$. In addition, the number of elements used in the finite element method is 40, which is also the same as the number used in Ref. [6]. For convenience of discussion, the dimensionless natural frequency $\bar{\omega}$ and the dimensionless fluid velocity $\bar{U}$ are defined by

$$\bar{\omega}_n = \omega_n R^2 \sqrt{\frac{m_p + m_f}{EI}}, \quad \bar{U} = UR\sqrt{\frac{m_f}{EI}}. \tag{30}$$

First, the static equilibrium positions of the fluid-conveying semi-circular pipe are examined for the three cases of the equations of motion. When the dimensionless fluid velocity is given by $\bar{U} = 4$, the displacements representing the static equilibrium position are depicted in Fig. 2, where Fig. 2(a) is for the radial displacement while Fig. 2(b) is for the circumferential displacement. The solid, dashed and dotted lines in Fig. 2 correspond to Cases I, II and III, respectively. These patterns of lines are used in the following figures in this section. As shown in Fig. 2, no difference is observed between the equilibrium positions of Cases II and III; however, the position of Case I is somewhat different from those of Cases II and III.

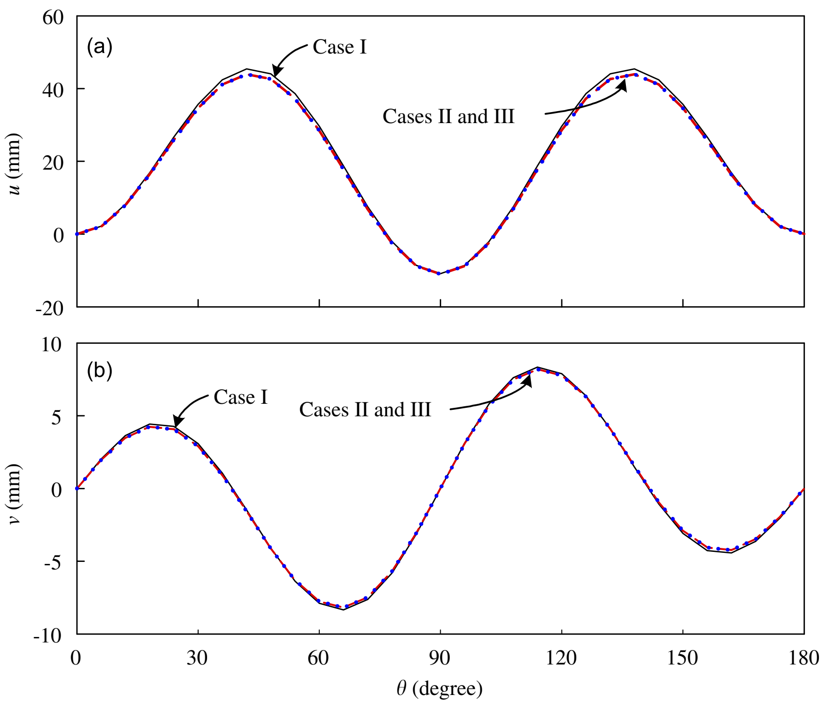

Fig. 2. Displacements representing the static equilibrium when $\bar{U} = 4$: (a) radial displacement; and (b) circumferential displacement Case I (solid line); Case II (dashed line); Case III (dotted line).

Next, the natural frequencies of the pipe are analysed to compare the numerical differences resulting from the three cases of modelling. For this purpose, the lowest three dimensionless natural frequencies for the variation of the dimensionless fluid velocity are illustrated in Fig. 3. The natural frequencies of the first and second modes show relatively large differences between Case I and the other cases, in Figs. 3(a) and (b). In contrast, the natural frequencies of the third mode, presented in Fig. 3(c), have no such large differences between the three cases. It is also seen in Figs. 3(a) and (b) that the natural frequency differences in the first and second modes increase with the fluid velocity. When the dimensionless fluid velocity is given by $\bar{U}=4$, the lowest three dimensionless natural frequencies are compared for the three cases in Table 2. The largest difference of 4.2% is shown in the first mode.

Finally, the differences of the mode shapes between the three cases are investigated. Fig. 4 shows the displacements representing the mode shapes for the lowest three natural frequencies when $\bar{U}=4$. In Fig. 4, the first and second columns correspond to the radial and circumferential displacements for the mode shapes, respectively, while the first, second and third rows correspond to the first, second and third modes. As illustrated in Fig. 4, even though the three cases do not have visible differences in the displacements of the first mode, they have considerable differences in the displacements of the second and third modes.

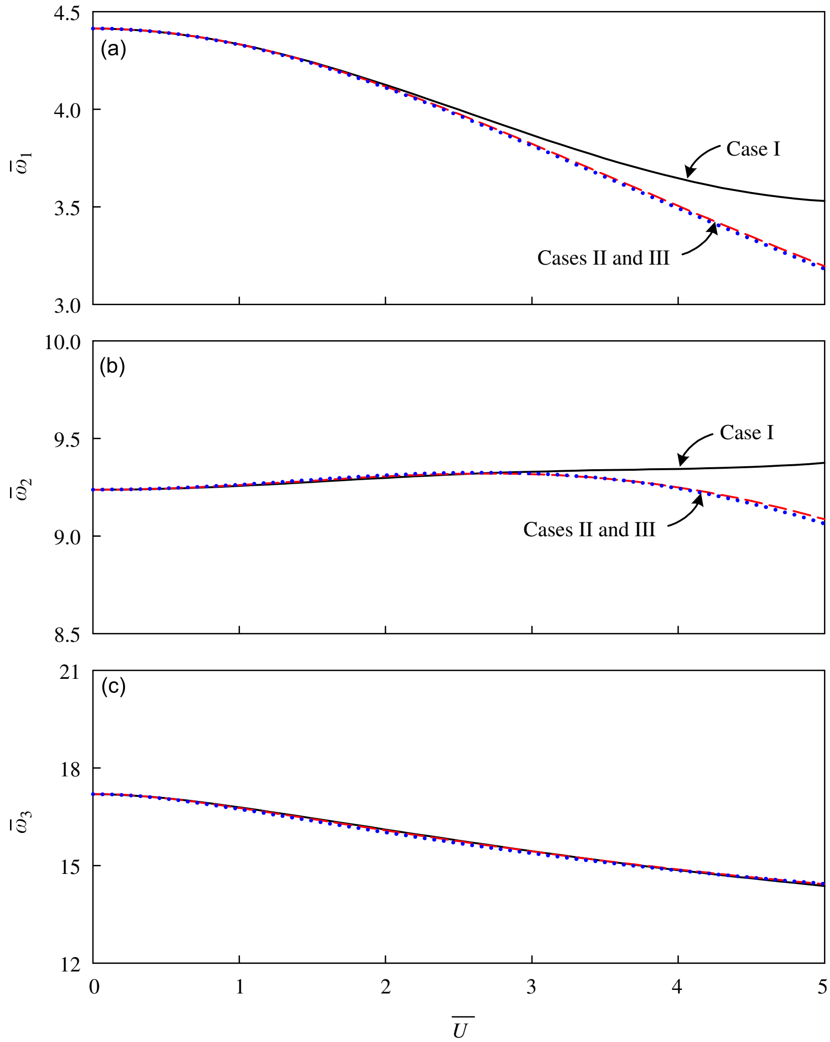

Fig. 3. Dimensionless natural frequencies for the variation of the dimensionless fluid velocity: (a) first mode; (b) second mode; and (c) third mode. Case I (solid line); Case II (dashed line); Case III (dotted line).

Table 2
Comparison of the dimensionless natural frequencies between the three cases when $\bar{U} = 4$

	First mode	Second mode	Third mode
Case I	3.64575	9.34364	14.85636
Case II	3.50402	9.24832	14.87272
Case III	3.49194	9.24305	14.84235
Difference between Cases I and II	−3.9%	−1.0%	0.1%
Difference between Cases I and III	−4.2%	−1.1%	−0.1%

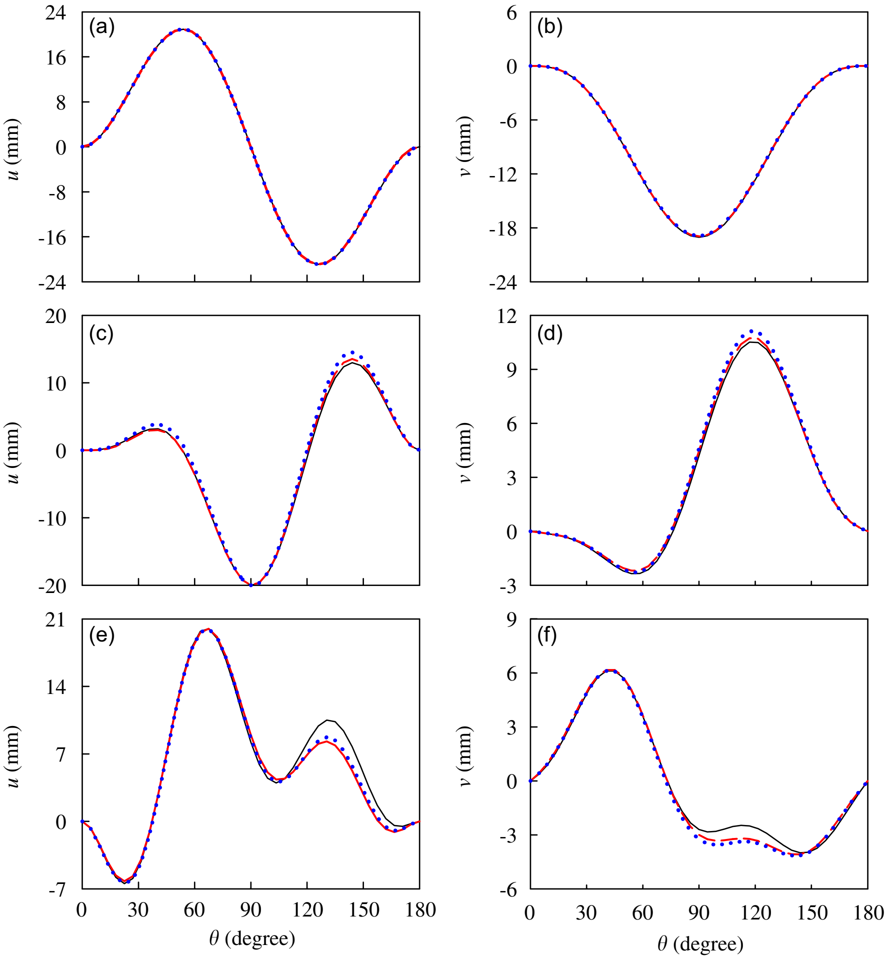

Fig. 4. Displacements representing the mode shapes when $\bar{U} = 4$: (a) radial displacement of the first mode; (b) circumferential displacement of the first mode; (c) radial displacement of the second mode; (d) circumferential displacement of the second mode; (e) radial displacement of the third mode; and (f) circumferential displacement of the third mode. Case I (solid line); Case II (dashed line); Case III (dotted line).

5. Summary and conclusions

In this study, the new fluid velocity expression was presented for the dynamic analysis of an extensible semi-circular pipe conveying fluid. The proposed fluid velocity expression was obtained from the material differentiation with respect to time. This fluid velocity expression is different from the previous velocity expression based on Love's kinematical relations. With the proposed velocity expression, the equations of in-plane motion for the clamped-clamped semi-circular pipe conveying fluid were derived by using the Hamiltonian approach.

To prove that the proposed fluid velocity expression is more reasonable than the previous velocity expression, the three cases of governing equations for the fluid-conveying semi-circular pipe were derived and examined. The equations corresponding to Cases I, II and III were derived by using the Hamiltonian approach with the proposed velocity, the Newtonian approach with the previous velocity, and the Hamiltonian approach with the previous velocity, respectively. The theoretical investigation showed that the presented velocity expression resulted in more acceptable equations of motion than the previous velocity. In addition, it was observed from the numerical investigation that the equilibrium positions, the natural frequencies and the mode shapes have non-negligible differences between the three cases.

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